

# Coupling between Majorana fermions and Nambu-Goldstone bosons inside a non-Abelian vortex in dense QCD

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## Abstract

Quark matter at high density may exhibit color superconductivity. As magnetic flux tubes in metallic superconductors, color magnetic flux tubes as non-Abelian vortices appear in the color-flavor locked phase of high density QCD.  $CP^2$  Nambu-Goldstone bosons and Majorana fermions belonging to the triplet representation are known to be localized around a non-Abelian vortex. In this paper, we determine the coupling of these bosonic and fermionic modes by using the nonlinear realization method.

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## I. INTRODUCTION

It is an interesting question what a state of matter becomes at extremely high density such as the cores inside neutron stars. At very high densities, the quarks are expected to form Cooper pairs to show color superconductivity [1, 2] (see Refs. [3, 4] as a review). At intermediate densities the up and down quarks participate in condensation to form the two-flavor superconducting (2SC) phase. The color-flavor locked (CFL) phase, where all three flavors (up, down and strange quarks) participate in condensations, is expected to be realized at asymptotically high densities. In this phase the  $SU(3)_C \times SU(3)_L \times SU(3)_R$  symmetry is broken down to  $SU(3)_{C+L+R}$  [2, 3, 5]. The gap function in the CFL phase is given by  $\Delta^{\alpha i} \propto \epsilon^{\alpha\beta\gamma} \epsilon^{ijk} \langle \psi^{\beta j} \psi^{\gamma k} \rangle$ , where  $i, j, k$  are color indices, and  $\alpha, \beta, \gamma$  are flavor indices. As the magnetic flux tubes can be introduced in a type-II metallic superconductor in the presence of a magnetic field, the color magnetic flux tubes are present stably in the CFL phase as well [6–9]; see Ref. [10] as a review. In the framework of the Ginzburg-Landau approach [11–13], the detailed profile of the gap function was calculated in Ref. [9]. An important consequence of non-Abelian vortices is that, the color-flavor-locked symmetry  $SU(3)_{C+L+R}$  of the ground state CFL phase is spontaneously broken down to its subgroup  $SU(2)_{C+L+R} \times U(1)_{C+L+R}$  in the presence of a non-Abelian vortex. This symmetry breaking generates the Nambu-Goldstone bosons around the vortex, which parametrize a coset space  $\mathbb{CP}^2 \simeq SU(3)_{C+L+R} / [SU(2)_{C+L+R} \times U(1)_{C+L+R}]$  [7, 14]. Accordingly, the vortex equations allow a continuous family of degenerate solutions, corresponding to  $\mathbb{CP}^2$ . These modes are known as the “internal orientational zero modes” and each point of  $\mathbb{CP}^2$  corresponds to the direction of color flux which the non-Abelian vortex carries. These  $\mathbb{CP}^2$  modes are localized around the vortex and it is possible to construct a two-dimensional  $(t, z)$  effective action on the vortex world sheet by integrating the original action over the coordinates  $(x, y)$  perpendicular to vortex direction along the  $z$ -axis [15, 16]. The interactions of the localized  $\mathbb{CP}^2$  modes with the gluons [17] and photons [18, 19] in the bulk were studied before. When the CFL phase is rotating, first,  $U(1)$  superfluid vortices are created [20] as liquid helium superfluids. Then, the superfluid vortex decays into three color flux tubes [21] and they will constitute a non-Abelian vortex lattice. A lattice of non-Abelian vortices shows a color ferromagnetism [22]. Recently, it has also been shown that a non-Abelian vortex has the Aharanov-Bohm phase [23].

The above results were derived based on the Ginzburg-Landau effective theory, which is a macroscopic theory valid only at the scale larger than the penetration depth and coherence length and at temperatures close to the transition temperature. In order to figure out the whole structure of the non-Abelian vortices in the color superconducting phase, including the region inside their cores, it is necessary to consider fermion dynamics from the Bogoliubov-de Gennes (BdG) equation. By solving the BdG equations with the given vortex gap profile, it is possible to find fermion zero modes in the center of the vortex. In the case of a non-Abelian vortex, Majorana fermion zero modes belonging to the triplet of  $SU(2)_{C+F}$  were found in Ref. [24] that are also in agreement with the index theorem [25]. One remarkable consequence of such Majorana fermion zero modes is that vortices obey non-Abelian statistics [26] as the case of chiral  $p$  wave superconductors [27], if vortices are parallel or confined in two-dimensional plane.

Questions arising immediately are whether fermion zero modes localized around the vortex interact with  $\mathbb{CP}^2$  Nambu-Goldstone bosons that are also localized around the vortex, and if so, how they interact and how one can derive the effective action to describe their

interaction. A drawback of the BdG formalism is that it is not easy to deal with bosonic modes such as  $\mathbb{C}P^2$  modes that we are considering. To overcome this problem, we use the nonlinear realization method that was introduced for the first time in particle physics long ago [28–30], in the context of chiral symmetry breaking while describing the dynamics of various interacting pion fields. It is well known today that symmetries play a significant role in determining the low-energy interactions of massless particles. When symmetries are broken spontaneously by the ground state of the system, massless Nambu-Goldstone bosons appear in this process. The scattering amplitudes of these Nambu-Goldstone modes are uniquely determined by certain low-energy theorems[31, 32]. The easiest way to understand the implications of low-energy theorems is to construct a low-energy effective action in terms of nonlinearly transforming Nambu-Goldstone bosons under the broken symmetry group. The generic framework of nonlinear realization is in general used to construct a low-energy effective action [29].

In this paper we use the nonlinear realization method to construct the effective action to describe the coupling between the triplet Majorana fermions and  $\mathbb{C}P^2$  Nambu-Goldstone bosons inside a non-Abelian vortex in the CFL phase of dense QCD.

The paper is organized as follows. In Sec. II, we discuss the fermion zero mode solutions of the BdG equations that transform under the low-energy group in the presence of a generic vortex profile function (see [24, 25] for a more detailed derivation). We divide the Sec. III in two subsections. In the first part, Sec. III.1, we recall the general framework of nonlinear realization of a compact, connected, semisimple Lie group  $G$  which is spontaneously broken down to a specified subgroup  $H$  and in Sec. III.2 we apply the nonlinear realization method to the two-dimensional effective action for the fermion zero modes in the presence of the  $\mathbb{C}P^2$  bosonic zero modes. Section IV is devoted to summary and discussion.

## II. FERMIONIC ZERO MODES AND EFFECTIVE ACTION

In this paper, our intention is to study the nonlinear realization due to the action of the color-flavor locked symmetry  $SU(3)_{C+F}(SU(3)_{C+L+R})$  on the low-energy massless fields in the CFL phase of dense QCD and derive the effective action of these modes. To do so, we start with the BdG equation and it can be determined from the generic Hamiltonian with a given pairing  $\text{gap}(\Delta)$  of superconductivity,

$$\mathcal{H} = \bar{\Psi}_i^\alpha \left( \hat{\mathcal{H}}_0 \delta_{ij} \delta^{\alpha\beta} + \tilde{\Delta}_{ij}^{\alpha\beta} \right) \Psi_j^\beta \quad (1)$$

where  $i, j = \{r, g, b\}$ ,  $\alpha, \beta = \{u, d, s\}$ .  $\hat{\mathcal{H}}_0$  and  $\tilde{\Delta}_{ij}^{\alpha\beta}$  are defined as

$$\hat{\mathcal{H}}_0 = \begin{pmatrix} -i\vec{\gamma} \cdot \vec{\nabla} - \gamma^0 \mu & 0 \\ 0 & -i\vec{\gamma} \cdot \vec{\nabla} + \gamma^0 \mu \end{pmatrix}, \quad \tilde{\Delta}_{ij}^{\alpha\beta} = \begin{pmatrix} 0 & \Delta_{ij}^{\alpha\beta} \gamma^5 \\ -\Delta_{ij}^{*\alpha\beta} \gamma^5 & 0 \end{pmatrix}. \quad (2)$$

Here  $\Delta_{ij}^{\alpha\beta}$  is the generic gap function, and  $\Psi$  is the fermion written in the Nambu-Gor'kov basis as

$$\Psi = \begin{pmatrix} \psi \\ \psi_c \end{pmatrix}, \quad \psi_c = e^{i\eta_c} i\gamma^2 \psi^*, \quad (3)$$

where  $\eta_c$  is an arbitrary phase and  $\psi$  is defined as a  $3 \times 3$  matrix whose elements consist of quarks with color and flavor indices,

$$\psi_i^\alpha = (\psi)_{\alpha i} = \begin{pmatrix} u_r & u_g & u_b \\ d_r & d_g & d_b \\ s_r & s_g & s_b \end{pmatrix}. \quad (4)$$

In this paper we work in the Nambu-Gor'kov basis where  $\psi$  and  $\psi_c$  are treated as particle and hole. This Hamiltonian density (1) leads us to the BdG eigenvalue equation,

$$\mathcal{H}\Psi = \mathcal{E}\Psi. \quad (5)$$

We assume the form of the gap function to be

$$\Delta_{ij}^{\alpha\beta} = \epsilon^{\alpha\beta\gamma} \epsilon_{ijk} \Delta_\gamma^k. \quad (6)$$

The transformation properties of the gap function under the action of  $(e^{i\theta}, U_c, U_f) \in U(1)_B \times SU(3)_C \times SU(3)_F$  can be expressed as

$$\Delta_\gamma^k \rightarrow e^{i\theta} (U_c)^\alpha_\gamma \Delta_\alpha^i (U_f^T)_i^k. \quad (7)$$

At the CFL phase ground state, the gap function takes the form of

$$\Delta_\gamma^k = \Delta_{\text{CFL}} \mathbf{1}_3 \quad (8)$$

where  $\Delta_{\text{CFL}}$  is a constant. This gap function breaks the  $U(1)_B \times SU(3)_C \times SU(3)_F$  symmetry and only the diagonal symmetry group, the so-called color-flavor locking symmetry  $SU(3)_{C+F}$ , is preserved.

In the presence of a vortex, the situation changes, because the gap function no longer remains constant. For the minimal winding non-Abelian vortices,  $\Delta_\gamma^k$  is given by

$$\Delta_\gamma^k = \text{diag}(\Delta_0, \Delta_0, \Delta_1), \quad (9)$$

where  $\Delta_1(r, \theta) = |\Delta_1|e^{i\theta}$  with the boundary conditions  $\Delta_0(\infty) = |\Delta_1(\infty)| = \Delta_{\text{CFL}}$  and  $|\Delta_0(0)|' = |\Delta_1(0)| = 0$ . A detailed profile of  $\Delta$  can be derived numerically by solving the BdG equation and the gap equation self-consistently, but it has yet to be done. This vortex carries a color magnetic flux confined inside its core (we have neglected it in the BdG equation). The CFL symmetry  $SU(3)_{C+F}$  is now broken spontaneously into  $SU(2)_{C+F} \times U(1)_{C+F}$  inside the vortex core. This generates Nambu-Goldstone bosonic zero modes which parametrize the  $\mathbb{CP}^2 \simeq SU(3)_{C+F}/[SU(2)_{C+F} \times U(1)_{C+F}]$  moduli space. The configuration in Eq. (9) corresponds to a particular point on  $\mathbb{CP}^2$ . These bosonic modes are gapped if one takes into account either the presence of the strange quark mass [16] or the nonperturbative quantum effect in the vortex world-sheet theory [33, 34].

In this section we are interested in writing down low-energy effective action in terms of fermion zero modes that transform under the unbroken  $SU(2)_{C+F}$ , discussed above. The explicit form of the zero modes can be found by solving the BdG equation (5). To do so let us expand  $\Psi$  in the  $U(3)$  generators as

$$\begin{aligned} \Psi_{\alpha i} &= \psi^a T_{\alpha i}^a, & \text{Tr}(T^a T^b) &= \delta^{ab}, \\ T^9 &= \frac{1}{\sqrt{3}} \mathbf{1}_3, & T^a &= \frac{1}{\sqrt{2}} \lambda^a, & \{a, b\} &= \{1, 2, \dots, 8\} \end{aligned} \quad (10)$$

where  $\lambda^a$  are the Gell-Mann matrices. Now the Hamiltonian (1) can be rewritten in this basis as

$$\mathcal{H} = \bar{\Psi}^a \left( \hat{\mathcal{H}}_0 \delta_{ab} + \Theta_{ab} \right) \Psi^b, \quad \{a, b\} = [1, 2 \dots, 9] \quad (11)$$

where

$$\Theta_{ab} = T_{i\alpha}^a \tilde{\Delta}_{ij}^{\alpha\beta} T_{\beta j}^b = \begin{pmatrix} -\tilde{\Delta}_1 & & & & & & & & \\ & \tilde{\Delta}_1 & & & & & & & \\ & & -\tilde{\Delta}_1 & & & & & & \\ & & & -\tilde{\Delta}_0 & & & & & \\ & & & & \tilde{\Delta}_0 & & & & \\ & & & & & 0 & & & \\ & & & & & & 0 & & \\ & & & & & & & \frac{1}{3}(\tilde{\Delta}_1 - 4\tilde{\Delta}_0) & \frac{\sqrt{2}}{3}(\tilde{\Delta}_1 - \tilde{\Delta}_0) \\ & & & & & & & \frac{\sqrt{2}}{3}(\tilde{\Delta}_1 - \tilde{\Delta}_0) & \frac{2}{3}(2\tilde{\Delta}_0 + \tilde{\Delta}_1) \end{pmatrix} \quad (12)$$

and the equations for the triplet states can be written as

$$\left( \hat{\mathcal{H}}_0 - \tilde{\Delta}_1 \right) \Psi^1 = 0, \quad \left( \hat{\mathcal{H}}_0 + \tilde{\Delta}_1 \right) \Psi^2 = 0, \quad \left( \hat{\mathcal{H}}_0 - \tilde{\Delta}_1 \right) \Psi^3 = 0. \quad (13)$$

The equations for singlet states are

$$\begin{pmatrix} \hat{\mathcal{H}}_0 + \frac{1}{3}(\tilde{\Delta}_1 - 4\tilde{\Delta}_0) & \frac{\sqrt{2}}{3}(\tilde{\Delta}_1 - \tilde{\Delta}_0) \\ \frac{\sqrt{2}}{3}(\tilde{\Delta}_1 - \tilde{\Delta}_0) & \hat{\mathcal{H}}_0 + \frac{2}{3}(2\tilde{\Delta}_0 + \tilde{\Delta}_1) \end{pmatrix} \begin{pmatrix} \Psi^8 \\ \Psi^9 \end{pmatrix} = 0. \quad (14)$$

Following Eq. (4) the triplet can be expressed as

$$\psi^1 = \frac{d_r + u_g}{\sqrt{2}}, \quad \psi^2 = \frac{d_r - u_g}{i\sqrt{2}}, \quad \psi^3 = \frac{u_r - d_g}{\sqrt{2}}, \quad (15)$$

while the singlets are identified as

$$\psi^8 = \frac{u_r + d_g - 2s_b}{\sqrt{6}}, \quad \psi^9 = \frac{u_r + d_g + s_b}{\sqrt{3}}. \quad (16)$$

As was shown in Refs. [24, 25], the singlet mode is non-normalizable. Here we are only concerned with the triplet zero modes as low-energy excitations. The triplet zero mode quarks are analytically found by solving Eq. (13) by expressing fermions as

$$\begin{aligned} \Psi_R^{(1)}(r, \theta) &= C_1 \begin{pmatrix} \varphi_R(r, \theta) \\ \eta_R(r, \theta) \end{pmatrix}, \quad \Psi_R^{(2)}(r, \theta) = C_2 \begin{pmatrix} \varphi_R(r, \theta) \\ -\eta_R(r, \theta) \end{pmatrix}, \quad \Psi_R^{(3)}(r, \theta) = C_3 \begin{pmatrix} \varphi_R(r, \theta) \\ \eta_R(r, \theta) \end{pmatrix} \\ \Psi_L^{(1)}(r, \theta) &= C'_1 \begin{pmatrix} \varphi_L(r, \theta) \\ \eta_L(r, \theta) \end{pmatrix}, \quad \Psi_L^{(2)}(r, \theta) = C'_2 \begin{pmatrix} \varphi_L(r, \theta) \\ -\eta_L(r, \theta) \end{pmatrix}, \quad \Psi_L^{(3)}(r, \theta) = C'_3 \begin{pmatrix} \varphi_L(r, \theta) \\ \eta_L(r, \theta) \end{pmatrix} \end{aligned} \quad (17)$$

where  $C_i$  and  $C'_i$  are normalization constants. The explicit solutions of  $\varphi_{R/L}$ (particle) and  $\eta_{R/L}$ (hole) can be found to be

$$\begin{aligned}\varphi_R(r, \theta) &= e^{-\int_0^r |\Delta_1(r')| dr'} \begin{pmatrix} J_0(\mu r) \\ iJ_1(\mu r) e^{i\theta} \end{pmatrix}, \quad \eta_R(r, \theta) = e^{-\int_0^r |\Delta_1(r')| dr'} \begin{pmatrix} J_1(\mu r) e^{-i\theta} \\ -iJ_0(\mu r) \end{pmatrix} \\ \varphi_L(r, \theta) &= e^{-\int_0^r |\Delta_1(r')| dr'} \begin{pmatrix} J_0(\mu r) \\ -iJ_1(\mu r) e^{i\theta} \end{pmatrix}, \quad \eta_L(r, \theta) = e^{-\int_0^r |\Delta_1(r')| dr'} \begin{pmatrix} -J_1(\mu r) e^{-i\theta} \\ -iJ_0(\mu r) \end{pmatrix},\end{aligned}\tag{18}$$

for the generic shape of the vortex profile  $|\Delta_1(r)|$ , where  $J_{0,1}$  are the Bessel functions. It should be noticed that the zero mode solutions written above satisfy the following ‘‘Majorana condition’’

$$\Psi_{R/L} = \kappa \gamma^2 \Psi_{R/L}^* \tag{19}$$

where  $\kappa = \pm 1$  for right and left modes, respectively.

As we have seen that the zero modes are eigenstates of the operator  $(\hat{\mathcal{H}}_0 \delta_{ab} + \Theta_{ab})$ , so the effective action that contains only zero modes may be written as

$$\mathcal{L}_{\text{eff}} = \int d^2x \bar{\Psi}(t, z, x, y) (-i\gamma^I \partial_I) \Psi(t, z, x, y), \quad I = \{0, 3\}. \tag{20}$$

The  $z$  dependence enters here in a factorized way:

$$\Psi_L^a(t, z, x, y) = \chi_L^a(t, z) \Psi_{0L}^a(x, y), \quad \Psi_R^a(t, z, x, y) = \chi_R^a(t, z) \Psi_{0R}^a(x, y). \tag{21}$$

where  $a = \{1, 2, 3\}$  (no summation over  $a$ ). The two-dimensional spinors  $\chi^a(t, z)$  are defined by

$$\chi^a(t, z) = \begin{pmatrix} \chi_L^a(t, z) \\ \chi_R^a(t, z) \end{pmatrix}. \tag{22}$$

Using the normalization condition  $\int d^2x \Psi_0^\dagger(x, y) \Psi_0(x, y) = 1$ , the effective action can be derived in terms of the two-dimensional spinor  $\chi^a(t, z)$  as

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= -i\chi_a^\dagger (\dot{\chi}^a - v(\mu, \Delta) \partial_z \chi^a) \\ &= -i\text{Tr} \chi^\dagger (\dot{\chi} - v(\mu, \Delta) \partial_z \chi), \quad \chi = \chi^a \tau^a\end{aligned}\tag{23}$$

where  $v(\mu, \Delta)$  is the velocity defined by

$$v_{L/R}(\mu, \Delta) = \int d^2x \Psi_{0L/R}^\dagger \gamma^0 \gamma^3 \Psi_{0L/R}. \tag{24}$$

This can be expanded as

$$v_{L/R}(\mu, \Delta) = \int d^2x [\psi_{L/R}^\dagger(r, \theta) \sigma^3 \psi_{L/R}(r, \theta) - \psi_{cL/R}^\dagger(r, \theta) \sigma^3 \psi_{cL/R}(r, \theta)] \tag{25}$$

where  $\psi$  and  $\psi_c$  are particle and hole solutions, respectively. The exact value of  $v_{L/R}(\mu, \Delta)$  depends on the exact profile of the vortex. This can be noted if we write the  $v_{L/R}(\mu, \Delta)$  explicitly as

$$v_{\text{fermi}} \equiv v_L(\mu, \Delta) = v_R(\mu, \Delta) = \int d^2x [\varphi^\dagger(r, \theta) \sigma^3 \varphi(r, \theta) - \eta^\dagger(r, \theta) \sigma^3 \eta(r, \theta)]. \quad (26)$$

The fermion zero modes shown above are excitations around the particular vortex configuration in Eq. (9). If we consider a configuration having the winding number in the second or third diagonal component, we have fermion zero modes in a different block. They all correspond to the particular points of the  $\mathbb{CP}^2$  moduli space. The general vortex configurations correspond to the  $\mathbb{CP}^2$  moduli space, and by considering them all together one can discuss the interaction between the fermion zero modes and the  $\mathbb{CP}^2$  Nambu-Goldstone bosons, which is the main purpose of this paper discussed in the next section.

### III. NONLINEAR REALIZATION IN THE CFL PHASE

In the first subsection, we would like to summarize briefly the nonlinear realization of a compact, connected, semisimple Lie group  $G$  which is spontaneously broken down to a specified subgroup  $H$ . Now let us consider a situation where we can introduce fields transforming linearly only under the unbroken subgroup  $H$  of the full group  $G$ . In this system, it is possible to classify all possible nonlinear realizations of the symmetry group  $G$  that become linear under the action of the unbroken subgroup  $H$  [29]. In the second subsection we analyze the nonlinear realization for the CFL phase where  $\text{SU}(3)_{\text{C+F}}$  symmetry is spontaneously broken down to  $\text{U}(1) \times \text{SU}(2)$  by the presence of a vortex and derive the effective action in terms of a Nambu-Goldstone mode and fermion zero modes.

#### III.1. Nonlinear realization

Let us introduce a field  $\Psi$  which transforms under a linear(preferably irreducible) representation of  $G$  as

$$\Psi' = D(g)\Psi, \quad g \in G. \quad (27)$$

Now let us suppose that the system goes through a spontaneous symmetry breaking that leaves  $H$ , a continuous subgroup of  $G$ , unbroken. We also assume that the basis has been chosen in order to represent  $D(h)(h \in H)$  in a block diagonal form. To derive a nonlinear realization, we may define a “reducing matrix,”  $L_{\phi_{\alpha\beta}}$ , the elements of which are field variables (Nambu-Goldstone bosons). The properties of  $L_\phi$  can be summed up as follows:

- The matrix  $L_\phi$  belongs to the group  $G$ . For any finite-dimensional representation representation  $D(g)$ , the  $D(L_\phi)$  should be well defined.
- The matrix  $L_\phi$  transforms as

$$L'_\phi = g L_\phi h^{-1}, \quad g \in G, \quad h \in H. \quad (28)$$

i.e. the columns of the reducing matrix transform among themselves according to some representation of  $H$ . It follows from Eq. (28) that for any finite-dimensional representation,  $D(L_\phi)$  transforms as

$$D(L_\phi) \rightarrow D(gL_\phi h^{-1}) = D(g)D(L_\phi)D(h^{-1}). \quad (29)$$

- The subgroup  $H$  operates on the reducing matrix  $L_\phi$  linearly as

$$L'_\phi = hL_\phi h^{-1}, \quad h \in H. \quad (30)$$

These properties of the reducing matrix help us to represent the nonlinear realization from the linear one by defining a new field,

$$\psi = D(L_\phi^{-1})\Psi. \quad (31)$$

The transformation properties of  $\psi$  can be derived as

$$\psi' = D(h_{\phi,g})D(L_\phi^{-1})D(g)D(g^{-1})\Psi = D(h_{\phi,g})\psi. \quad (32)$$

Here  $h_{\phi,g}$  in general has a nonlinear structure; that is, it depends upon the fields ( $\phi$ ) that parametrize  $L_\phi$ . However, the derivative of  $\psi$  does not transform covariantly and this problem can be solved by introducing a covariant derivative as

$$\mathcal{D}_\mu \psi = \partial_\mu \psi + ie_\mu^a D(h_a) \psi, \quad (33)$$

where  $e_\mu^a h_a$  would behave as a gauge field and it is related to the Maurer-Cartan one-form as

$$-iL_\phi^{-1}\partial_\mu L_\phi = e_\mu^a h_a + e_\mu^\alpha T_\alpha, \quad (34)$$

where  $h^a$  and  $T^a$  are the generators of  $H$  and  $G/H$ , respectively.

The transformation properties of the covariant derivative, defined above, can be derived as follows:

$$\begin{aligned} \mathcal{D}_\mu \psi' &= \mathcal{D}_\mu (D(h_{\phi,g})D(L_\phi^{-1})\Psi) \\ &= \partial_\mu (D(h_{\phi,g})D(L_\phi^{-1})\Psi) + ie_\mu^a D(h_a) (D(h_{\phi,g})D(L_\phi^{-1})\Psi) \\ &= D(h_{\phi,g}) (\partial_\mu + ie_\mu^a D(h_a)) \psi, \end{aligned} \quad (35)$$

where  $e_\mu^a D(h_a)$  is defined by the inhomogeneous gauge transformation as

$$e_\mu^a D(h_a) = e_\mu^a D(h_{\phi,g}^{-1})D(h_a)D(h_{\phi,g}) - iD(h_{\phi,g}^{-1})\partial_\mu D(h_{\phi,g}). \quad (36)$$



### III.2. Nonlinear realization inside a non-Abelian vortex

In this subsection we demonstrate the nonlinear realization for the case of fermionic zero modes, discussed in Sec. II and derive the two-dimensional effective action. We consider here the nonlinear realization of the broken  $SU(3)_{C+F}(G)$  symmetry which becomes linear under unbroken  $SU(2)_{C+F} \times U(1)_{C+F}(H)$  color-flavor transformation. This symmetry breaking generates Nambu-Goldstone modes which parametrize the coset space  $G/H \simeq \mathbb{CP}^2$ . According to Sec. III.1, the nonlinear realization introduces covariant derivatives in the effective action. So the new effective action describe the interaction between the fermionic zero modes and  $\mathbb{CP}^2$  Nambu-Goldstone bosons via the covariant derivative. In order to establish the nonlinear realization in this case, we first need to construct the reducing matrix  $L_{\phi i\alpha} \in SU(3)_{C+F}$ .

For the purpose of calculation we choose the basis such that the unbroken  $SU(2)_{C+F}$  subgroup lives in the upper left  $2 \times 2$  block of a  $3 \times 3$  matrix of the full symmetry group  $SU(3)_{C+F}$ . The fermionic zero mode  $\chi$ , as discussed in Sec. II, transforms as the adjoint representation of the symmetry group  $SU(2)_{C+F}$ . So the action of  $SU(3)_{C+F}$  on  $\chi^a$ , which transforms linearly under the unbroken  $SU(2)_{C+F}$ , would generate a nonlinear realization by introducing the  $\mathbb{CP}^2$  Nambu-Goldstone modes  $\phi$  into the effective action.

As described in Refs. [29, 35], the simplest way to parametrize the reducing matrix is to write it as

$$U(\phi) = e^{i\phi \cdot T}, \quad \text{where } T \in G/H. \quad (37)$$

Following this we choose the form of the reducing matrix to be

$$U(\phi) = e^{if(|\phi|)A}, \quad (38)$$

where  $A$ ,  $\hat{n}$  and  $f(|\phi|)$  are defined as

$$A = \begin{pmatrix} \mathbf{0}_2 & \hat{n} \\ \hat{n}^\dagger & 0 \end{pmatrix}, \quad \hat{n} = \frac{1}{\sqrt{|\phi^1|^2 + |\phi^2|^2}} \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix}, \quad f(|\phi|) = \arctan |\phi|. \quad (39)$$

The explicit expression for the reducing matrix  $U$  can be written as

$$U = \begin{pmatrix} U_2 & iU^- \\ iU^+ & U_0 \end{pmatrix} = \mathbf{1}_3 - 2\sin^2 \frac{f}{2} A^2 + i\sin f A. \quad (40)$$

In the above equation,  $U_2$  is a  $2 \times 2$  matrix,  $U^-$  is a column,  $U^+ = U^{-\dagger}$  and  $U_0$  is a number. The components of the unitary matrix  $U$  are related by the eigenvalue equation,

$$U_2 U^- = U_0 U^-. \quad (41)$$

The explicit expressions of the components can be written as

$$\begin{aligned} U_2 &= \mathbf{1}_2 - 2\sin^2 \frac{f(|\phi|)}{2} \hat{n} \hat{n}^\dagger, & U^- &= \sin f(|\phi|) \hat{n} \\ U^+ &= \sin f(|\phi|) \hat{n}^\dagger, & U_0 &= \cos f(|\phi|). \end{aligned} \quad (42)$$

The homogeneous coordinates of the  $\mathbb{CP}^2$  coset space can be expressed in terms of the above coordinates as

$$Z = \begin{pmatrix} iU^- \\ U_0 \end{pmatrix} = \begin{pmatrix} i\sin f(|\phi|) \hat{n} \\ \cos f(|\phi|) \end{pmatrix} = \frac{1}{\sqrt{1 + |\phi|^2}} \begin{pmatrix} i\phi_1 \\ i\phi_2 \\ 1 \end{pmatrix}. \quad (43)$$

The covariant derivative can be determined by the Maurer-Cartan one-form,

$$-iU^\dagger \partial_I U = -i \begin{pmatrix} U_2 \partial_I U_2 + U^- \partial_I U^+ & iU_2 \partial_I U^- - iU^- \partial_I U_0 \\ -iU^+ \partial_I U_2 + iU_0 \partial_I U^+ & U_0 \partial_I U_0 + U^+ \partial_I U^- \end{pmatrix}, \quad (44)$$

where  $I = \{0, 3\}$ . This matrix belongs to the  $SU(3)$  Lie algebra as described in Eq. (34). So, to construct the covariant derivative defined in Eq. (33), we need to separate out the part which belongs to the  $SU(2)$  subalgebra. Following the above equation (44) and using the parametrization of Eq. (42), we can express  $e_I^a(\phi) \tau^a$  as

$$\begin{aligned} A_I(\phi) &= e_I^a(\phi) \tau^a \\ &= -i \left[ (U_2 \partial_I U_2 + U^- \partial_I U^+) - \frac{1}{2} \text{Tr}(U_2 \partial_I U_2 + U^- \partial_I U^+) \mathbf{1}_2 \right] \\ &= -i \left[ \{1 - \cos f(|\phi|)\} \left\{ \hat{n} \partial_I \hat{n}^\dagger - (\partial_I \hat{n}) \hat{n}^\dagger + [1 - \cos f(|\phi|)] (\hat{n}^\dagger \partial_I \hat{n}) \hat{n} \hat{n}^\dagger \right\} \right. \\ &\quad \left. + \frac{1}{2} \sin^2 f(|\phi|) \hat{n}^\dagger (\partial_I \hat{n}) \mathbf{1}_2 \right]. \end{aligned} \quad (45)$$

For small perturbations in the  $\mathbb{CP}^2$  space the approximate form of  $A_I(\phi)$  becomes

$$A_I(\phi) \simeq \begin{pmatrix} \Im[\phi_1^* \partial_I \phi_1 - \phi_2^* \partial_I \phi_2] & -i[\phi_1 \partial_I \phi_2^* - \phi_2^* \partial_I \phi_1] \\ -i[\phi_2 \partial_I \phi_1^* - \phi_1^* \partial_I \phi_2] & -\Im[\phi_1^* \partial_I \phi_1 - \phi_2^* \partial_I \phi_2] \end{pmatrix}. \quad (46)$$

So the covariant derivative can be written as

$$\begin{aligned} \mathcal{D}_I \chi(t, z) &= \partial_I \chi(t, z) + i[A_I(t, z), \chi(t, z)]. \\ \chi(t, z) &= \chi^a(t, z) \tau^a, \quad \tau^a \in su(2), \end{aligned} \quad (47)$$

where  $\chi^a$  are the triplet two-component fermionic zero modes, as described above. The final form of the fermionic effective action can be expressed as

$$\mathcal{L}_{\text{fermi}} = -i \text{Tr} [\chi^\dagger(t, z) \{ \mathcal{D}_0 \chi(t, z) - v(\mu, \Delta) \mathcal{D}_z \chi(t, z) \}]. \quad (48)$$

Here  $v$  is found to be the same for all three triplet zero modes, and in a matrix form it can be written from Eq. (26) as

$$v = v_{\text{fermi}} \sigma_3. \quad (49)$$

Following Eqs. (43) and (44) we can write down the bosonic  $\mathbb{CP}^2$  effective action as

$$\mathcal{L}_{\mathbb{CP}^2} = C_0 \{ \partial_0 Z^\dagger \partial_0 Z + (Z^\dagger \partial_0 Z) Z^\dagger \partial_0 Z \} + C_3 \{ \partial_3 Z^\dagger \partial_3 Z + (Z^\dagger \partial_3 Z) Z^\dagger \partial_3 Z \}. \quad (50)$$

where  $Z^\dagger Z = 1$  as defined in Eq. (43) and the coefficients  $C_0$  and  $C_3$  are derived in [10, 14, 15] by using the explicit form of the vortex profile functions in the Ginzburg-Landau theory.

In order to see the interaction between fermionic and bosonic modes more explicitly, let us reform the effective action. Let us first rescale the space-time to remove  $v_{\text{fermi}}$  by setting

$$t \rightarrow t, \quad z \rightarrow v_{\text{fermi}} z. \quad (51)$$

This changes the covariant derivative as

$$v_{\text{fermi}} \mathcal{D}_z \rightarrow \mathcal{D}_z. \quad (52)$$

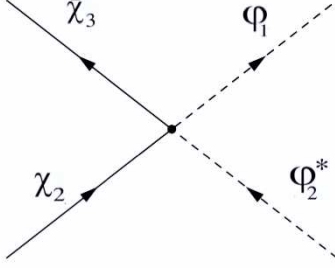


FIG. 1. The Feynman diagram for the effective coupling between fermion zero modes and  $\mathbb{CP}^2$  Nambu-Goldstone bosons.

We can also define effective two-dimensional gamma matrices as

$$\Gamma^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma^z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (53)$$

Then, the effective action can be rewritten in a Lorentz invariant way as

$$\mathcal{L}_{\text{eff}} = -i\text{Tr} \left[ \bar{\chi}(t, z) \not{D} \chi(t, z) \right], \quad \not{D} = \Gamma^I \mathcal{D}_I. \quad (54)$$

However this rescaling does not make the theory really Lorentz invariant because this rescaling is not consistent with the rescaling to make the  $\mathbb{CP}^2$  action (50) effectively Lorentz invariant.

The explicit form of the interaction can be expressed in terms of the components of the fields as

$$\text{Tr} \left[ \bar{\chi} [\not{A}(t, z), \chi(t, z)] \right] = i\epsilon^{abc} \bar{\chi}^a \not{A}^b \chi^c, \quad \not{A} = \Gamma^I A_I \quad (55)$$

where the components of  $\not{A}$  can be expressed from Eq. (45) as

$$\not{A}^1 = \Im[\phi_1 \not{\partial} \phi_2^* - \phi_2^* \not{\partial} \phi_1], \quad \not{A}^2 = \Re[\phi_1 \not{\partial} \phi_2^* - \phi_2^* \not{\partial} \phi_1], \quad \not{A}^3 = \Im[\phi_1^* \not{\partial} \phi_1 - \phi_2^* \not{\partial} \phi_2]. \quad (56)$$

The interaction between the Majorana fermions and Nambu-Goldstone modes obtained in the above implies the instability of these modes. For instance, two Majorana fermions can decay into two Nambu-Goldstone bosons, as shown in Fig. 1. We have not yet considered the strange quark mass. If the effect of the strange quark mass is taken into account, the effective potential of the  $\mathbb{CP}^2$  modes is present [16], which will suppress such a decay below the energy scale corresponding to the strange quark mass.

#### IV. SUMMARY AND DISCUSSION

In this paper we have derived the (1+1)-dimensional effective action of the Majorana fermion zero modes coupling to the  $\mathbb{CP}^2$  Nambu-Goldstone bosonic modes localized around a non-Abelian vortex in the CFL phase of dense QCD, by using the method of nonlinear realizations. These fermion zero modes arise in the vicinity of a non-Abelian vortex, so they live around the vortex core. We have integrated the action over the vortex codimensions

$(x, y)$  which leaves the zero modes as the main degrees of freedom on two-dimensional space (world sheet). The  $\mathbb{CP}^2$  Nambu-Goldstone modes are generated by the spontaneous breaking of  $SU(3)_{C+F}$  color-flavor symmetry of the CFL ground state to  $U(1)_{C+F} \times SU(2)_{C+F}$ . These modes appear inside the vortex core along with the fermion zero modes. To demonstrate their interaction we have used the nonlinear realization method. The fermion zero modes transform as the triplet representation of the unbroken group  $SU(2)_{C+F}$ , and this  $SU(2)_{C+F}$  also acts linearly on the  $\mathbb{CP}^2$  Nambu-Goldstone modes as an isotropy group. We have used the property of the nonlinear action of the full group  $SU(3)_{C+F}$  to generate the interaction term between the  $\mathbb{CP}^2$  Nambu-Goldstone modes and fermion zero modes by introducing a covariant derivative.

In this paper, we have studied only a single vortex. Around a multiple winding vortex, there appear, in general, localized Dirac fermion doublets of  $SU(2)$  [25] that also show non-Abelian statistics [36]. Since the nonlinear realization method is generally applied to arbitrary representation, it is straightforward to apply it to the doublet fermions in such cases.

It was shown in Refs. [33, 34] that a quantum mechanical potential is induced on the  $\mathbb{CP}^2$  modes once nonperturbative quantum effects are taken into account, and it implies the existence of confined monopoles and quark-hadron duality [34]. This potential may be changed or even eliminated if one takes into account fermion zero modes coupling to the  $\mathbb{CP}^2$  modes [37].

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